University of Houston

Math 6397

Mathematical Hemodynamics

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For:
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Part I

Partial Differential Equations

Chapter 1

Introduction

For the next few weeks, we will be introducing the mathematical theory of partial differential equations. To suggest that such a "theory" of partial differential equations exists (in the same sense that a theory of ordinary differential equations or a theory of functions of a single complex variable) is misleading. PDEs is a much larger subject than the two mentioned above (in fact, it includes both of them as special cases) and consequently a less well developed one. However, although a casual observer may decide the subject is simply a grab bag of unrelated techniques to handle different types of problems, there are certain themes that run throughout.

In order to illustrate these themes we are going to take two approaches. The first is to pose a group of questions that arise in many problems in PDEs (existence, multiplicity, etc.). As examples of different methods of attacking these problems, we examine some results from the theory of ODEs, advanced calculus and complex variables (with which we assume familiarity). The second approach is to examine three partial differential equations (Laplace's equation, the heat equation and the wave equation) in a very elementary fashion (again, this is probably a review for most of you). We will see that even the most elementary methods foreshadow deeper results found in the later portions of this course.

1.1 Basic Mathematical Questions

1.1.1 Existence

Questions of existence occur naturally throughout mathematics. The question of whether a solution exists *should* pop into a mathematician's head any time he or she writes an equation down. Appropriately, the problem of existence of solutions of partial differential equations occupies a large portion of this course, as we have seen. For now, we consider precursors of the PDE theorems to come.

Initial-value problems in ODEs

The prototype existence result in differential equations is for initial-value problems in ODEs.

Theorem 1.1.1. *(ODE Existence, Picard-Lindelöf) Let $D \subseteq \Re \times \Re^n$ be an open set, and let $\mathbf{F} : D \to \Re^n$ be continuous in its first variable and **uniformly Lipschitz** in its second; i.e. for $(t, y) \in D$, $\mathbf{F}(t, y)$ is continuous as a function of t, and there exists a constant γ such that $\forall (t, y_1)$ and $(t, y_2) \in D$ we have*

$$|\mathbf{F}(t, y_1) - \mathbf{F}(t, y_2)| \leq \gamma |y_1 - y_2| \tag{1.1.1}$$

*Then, for any $(t_0, y_0) \in D$ $\exists$ an interval $I \equiv (t^-, t^+)$ containing t_0 and at least one solution $y \in C^1(I)$ of the **initial value problem***

$$\frac{dy}{dt}(t) = \mathbf{F}(t, \mathbf{y}(t)) \tag{1.1.2}$$

$$\mathbf{y}(t_0) = \mathbf{y}_0 \tag{1.1.3}$$

The proof of this can be found in almost any text on ODEs. We make note of one version of the proof that is the source of many techniques in PDEs: the construction of an equivalent integral equation. In this proof, one shows that there is a continuous function $\mathbf{y}$ that satisfies

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{F}(s, \mathbf{y}(s))\, ds. \tag{1.1.4}$$

Then the fundamental theorem of calculus implies that $\mathbf{y}$ is differentiable and satisfies $(1.1.2), (1.1.3)$ (cf. the results on smoothness below). The solution of $(1.1.4)$ is obtained from an iterative procedure; i.e. we begin with an initial guess for the solution (usually the constant function $\mathbf{y}_0$) and proceed to calculate the value. This is known as Picard Iteration and we leave the details to you. Of course, to complete the proof one must show that this Picard iterative sequence does in fact converge to a solution.

We will see generalizations of this procedure used to solve PDEs in later lectures.