University of Houston

High School Mathematics Development Program

Understanding Algebra II

Author:
Bradshaw Stanley

Supervisor:
Martin Golubitsky

October 13, 2006

0.1 Preface

The goal of this document is to introduce high-school level students to the basic facts of Algebra II. It is unique insofar as it is one of the first texts to physically motivate most of the discussion of the mathematics. Thus, there may be a concern that in seeking to present a physical, "real-world" applications side of the subject that some of the precision and accuracy will be lost–this is indeed a valid concern, and oftentimes we will value clarity over absolute rigor.

The breakdown of the text is setup to accommodate the standard high school teacher's schedule: there are six Parts, each corresponding to one six-week period. Inside each Part is three chapters to be covered for that six-week period. Finally, each chapter is broken down into various, relatively short sections; the idea behind this is that the usual textbook often clutters the ideas with unnecessary verbiage and, indeed, overwhelms the student through winding pedantic discourses: the current text, to the contrary, gives taut, physically-motivated discussions of ideas that a student could easily review. Finally, each section is motivated by an initial question; for instance, Chapter 1.1 opens with the question, "What is a function?"

Unlike most high school courses, mathematics is a subject that is not solely dependent upon the quality of the teacher—it also depends on the willingness and ability of the student to go out on his own and practice. To this end, we have tried to include for every section at the end of each chapter a set of homework problems that a student can work on with a minimum amount of reliability upon his instructor; the purpose of this homework is two-fold: (1) it should be used by students to prove to themselves that the equations used and the claims made in the literature are valid, and (2) it should be used by instructors to ascertain the level of comprehension amongst the students. It is generally believed that knowledge of mathematics is not gained by a simple nod of recognition to the theory, but through actual practice and application of that theory. Since the problems at the end of the chapters have been designed to correlate strictly to the chapter-content, it is not advised that an instructor assign homework from a different text.

The guiding light behind the organization of this material is level of difficulty: it is hoped that through mastery of the basics in the first two chapters of any given Part, that more advanced motivations and applications can be witnessed in the third chapter. The same can be said for the problems given at the end of the chapters: the level of complexity increases as the students goes through them (while not escaping the material presented in the text). The goal is to show students—who often do not fully understand the power of mathematics—the way in which mathematics structures our very lives. This text, then, does not pick up with the usual Geometry course preceding it, in that it bypasses the Proof-Theorem-Definition approach used by most books for a less rigorous but more applications-friendly style of presentation.

Moving to a discussion of the content, it is hoped that this will be a largely self-contained exposition. The first two Parts are largely concerned with the algebra of Algebra II. The third and fourth Parts deal with the topology of Algebra II, and the fifth Part is a combination of the algebra and topology. The final part is an introduction to calculus. This is unusual since most textbooks try to simultaneously address the algebra and topology of each idea; however, it is believed by at least this author that this causes an unnecessary strain on the student to switch between algebraic ideas and topological ideas without either having been fully developed. The basic approach, then, is to "disassemble" the algebraic structure first, study the topological pieces, then reassemble and study the whole geometry.

Hence, Part 1 covers functional analysis and the appropriate operations one can perform to get solutions and physically-meaningful results from a function; Part 2 is a further exploration of more advanced solution methods with particular attention to linear algebra and systems; Part 3 is concerned with graphs of functions and trigonometry; Part 4–by far the most theoretical–deals with asymptotes and advanced trigonometric functions; Part 5 unifies the four preceding parts by exploring the complex number system, exponential and logarithmic functions, and series; Part 6 concludes the text by introducing the notions of continuity, limits, and differentiation. Part 6 may be entirely omitted if a teacher desires to concentrate more on the preceding parts.

As with all things, the student has to be kept in mind when writing to educate. Thus, all comments and suggestions regarding the development of this document are more than welcomed.

Brad Stanley
University of Houston
Bradshaw.Stanley@mail.uh.edu

Contents

Part I

First Six Weeks

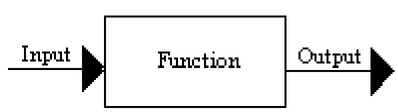
Chapter 1

Functional Analysis

We begin this chapter by defining a function. Comparing this introduction to Algebra II with what one finds in the typical Algebra II text (usually linear systems, etc.) will likely make this approach seem unorthodox. However, in order to motivate more serious ideas later on, it is important we understand functions first. Finally, the terminology and definitions presented here will be slightly different than what one finds in a standard textbook.

1.1 What is a function?

Most problems involving mathematics can be thought of as involving a machine. This machine, which we call a function, takes an input value, hums and whirls, and produces an output value.



This concept of the machine is very helpful, and oftentimes when solving complex problems, it will be useful to construct such a diagram. A question: where do these so-called "functions" arise?

- In computer science, users define commands for the computer to run. The computer then produces an output value, which can be anything from a text document to a graphic to running a calculation. The function here is determined by the computer and the programs being used.

- In chemical engineering, whenever a nuclear reactor is operating, one might wish to get a desired result (usually this is for safety reasons: for example, we don't want to have an explosion or a meltdown). In order for this to happen, we must be very careful about the input we give the reactor. The function here is highly complex and, indeed, very sensitive to the input values.

- In finance and business, when you open a checking account at a bank, you usually receive a certain percent of interest on the money you have put in. We can think of this starting money as an input value, then we can construct a function that will tell you the amount of money you will have accumulated over a given time frame.

- In biology, we can consider a vibrating string to be approximate to a vein in the human body. If we pluck this string, we can study the vibrations that result. A function relates the initial pluck of the string to the vibrations.

- In heavy machinery, it is important to consider the job to be done and the load the job requires on the machine. If the load is too high, then the machine could break down, tip over, or worse. The load that is applied and the result on the machine are related by a function.

As you can see, functions arise quite naturally in the physical world, and an understanding of them is key to understanding many fields. We can define a function as anything which takes an initial value and produces an output value. We then formalize this notion of a function as a mechanism, f, that takes an input value, x, and maps an output value, $f(x)$, using the given input value. Another way of writing this is, $f\colon x \to f(x)$, which is read as "f maps x onto $f(x)$." The word function can be used interchangeably with mapping or transformation. A quick example of a function is

$$f = x + 1$$

where it is easy to see that whenever our input value is two, $x = 2$, we have

$$f(x = 2) = f(2) = 2 + 1 = 3$$

Equivalently, we could just as easily compute

$$f(x = 5) = f(5) = 5 + 1 = 6$$

As the astute reader might notice, what has happened here is a substitution of the values 2 and 5 for the **variable**, x, in our function. Indeed, this substitution can go beyond simple numerical values, such that we could easily evaluate, say, $f(m)$. We do this by simply

$$f(m) = m + 1$$

where m can be either another function, or else a different variable. Further, functions do not have to be denoted solely by f, though it is the most common way to do so. We could just as easily have a function defined by g, or w, or whatever one might like—it is usually a matter of what will make the most sense and personal preference.

We can visually understand this as a graph,

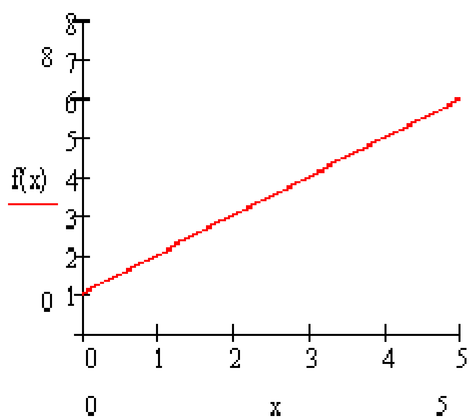


Figure 1.1: This graph shows the into mapping of f = x+1

which linearly maps our initial input, x (which can be determined by looking to the horizontal axis), into an output value, $f(x)$ (which can be determined by looking to the vertical axis). For instance, we see that when $x = 5$, $f(5) = 6$, or that when $x = 2$, $f(2)$ does, in fact, equal 3.

Before leaving this section, we need to get some basic terminology out of the way. We call the set of our input values, denoted by $\{x\}$, the **domain** of the function. We call the values which the function, f, can produce the **range**, $\{y\}$, of the function.

Another very important property of functions is that for every x in the domain there is *one and only one* value for the mapping $f(x)$. This means that, in the previous example, the mapping $f(5)$ *cannot* give two results, which we can easily verify. Functions which satisfy this condition are called **functional**.

Finally, we say that when a range, $\{y\}$, includes but is not limited to the values of $f(x)$, then a function, f, such that $f\colon x \to f(x)$ also maps x *into* y. Using the previous example, we see from the graph that the **range** is all the numbers from 0 to 8, or $y = 0...8$. We see that our domain is from 0 to 5, $x = 0...5$, and that our values from the function, f, are from 1 to 6, $f(x) = 1...6$. We can then say that our function, $f = x + 1$, maps x onto $f(x)$ and also *into* y. We then say that f is an into function with respect to y.

Further, whenever the range, $\{y\}$ is equal to the output, $f(x)$ (that is, $\{y\} = f(x)$), then we can say that $f\colon x \to f(x)$ and $f\colon x \to y$. Equivalently, we can say that f is an onto function with respect to y. Below is a graph of the same function, $f = x + 1$, but defined as an onto function; that is, the range is from 1 to 6 and the domain is from 0 to 5.

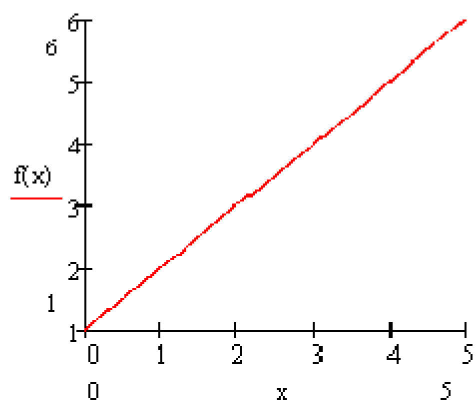


Figure 1.2: This graph shows the onto mapping of f = x+1

Whether a function maps its input values *into* or *onto* a range will become very important later on for determining the type of continuity a function has. For now, it is only important that you be able to identify when a function is into, onto, or neither. Problems at the end of this chapter can help you reassure yourself that you do in fact understand these concepts.

1.2 What kinds of functions are there?

For our purposes, there are two major classifications of functions: functions that are linear, and functions that are not. In Algebra I, you were largely exposed to linear equations and how to solve them. In Algebra II, we are generally not concerned with linear equations, and instead concentrate on what are known as "nonlinear" equations: these equations involve more complicated solution methods.

Let's quickly review basic algebra. We know how to solve an equation for the variable x of the form

$$x + 3 = 2$$

by subtracting 3 from both sides. This yields

$$x + 3 - 3 = 2 - 3$$

which shows us that

$$x = -1$$

This brings us to an important property of both linear and nonlinear equations: what we do to one side, we must also do to the other. This is nothing more than we already knew from Algebra I, and it is called the **Fundamental Property of Algebra**. To exemplify this idea one more time, we solve the equation of the familiar form,

$$\frac{2}{3}x = 3$$

We solve this equation by dividing both sides by $\frac{2}{3}$,

$$\frac{2}{3}x \div \frac{2}{3} = 3 \div \frac{2}{3} \Longrightarrow \frac{2}{3} \cdot \frac{3}{2}x = 3 \cdot \frac{3}{2}$$

which we know yields

$$x = \frac{9}{2}.$$

Just as we have referred to equations as linear or nonlinear, we may call functions linear or nonlinear. For instance, the function

$$f = 2x + 3$$

is a *linear* function, while

$$g = 2x^2 + 3$$

is *nonlinear*.

How do we determine if a function is linear or nonlinear? We do this by introducing what is known as an **operator** on the function. We denote this operator by ℓ.

In the 1960s at NASA, scientists and mathematicians dealt with functions which they could not fully define—that is, functions that could not be written down explicitly in terms of their variables. This meant the scientists and engineers could not be certain whether a given function was linear or nonlinear until experiments were run to determine linearity. The problem was that it could take weeks for any one experiment to be run, so the mathematicians and scientists often did what is known as "symbolic computations" (computations which do not involve any numbers, just variables). They used what they knew about linear and nonlinear functions to make predictions regarding shuttle flight. In fact, one famous prediction credited to J. Lorenz has turned out to be the governing prediction for all flight in outer-space. Lorenz arrived at his prediction by assuming the functions were nonlinear.

We define a linear function to be a function for which

$$\tilde{f}(\ell x) = \ell \cdot \tilde{f}(x).$$

where $\tilde{f}$ is the part of the function f that involves only the variable (that is, no constants are evaluated).

We can look to an example to verify this. Define a function $g\colon x \to g(x)$ such that $g = 5x - 4$; then by applying our definition of linearity, we see that $\tilde{g} = 5x$ implies

$$\tilde{g}(\ell x) = 5(\ell \cdot x) = \ell \cdot (5x) = \ell \cdot \tilde{g}(x)$$

Applying our definition to another transformation $f\colon x \to f(x)$ and $f = \frac{x^2}{2} + 9x$, we see that, in this case, $\tilde{f} = f$. Let's assume that f is linear, then $f(\ell x) = \ell \cdot f(x)$. In this case,

$$f(\ell x) = \frac{(\ell x)^2}{2} + 9(\ell x) = \frac{\ell^2 x^2}{2} + 9\ell x = \ell \cdot \left(\frac{\ell x^2}{2} + 9x\right) \neq \ell \cdot f(x),$$

which contradicts our linearity assumption. Note that even though the second term, $9x$, exhibits linearity, the function *overall* does not—this is what is important. Thus f is a nonlinear mapping.

This is all fine and well, since the linearity or the *degree of nonlinearity* of a function can give us a qualitative understanding of our solutions, as we shall see later. However, before discussing these behaviors, we are now presented with another problem: what if we only have a desired output and do not know our input? That is, what if we only know our range and not the domain? These kinds of questions become important, for instance, when one is trying to manufacture sweaters. If we know that we want to produce a certain number of sweaters every hour that varies with the hour, how much fabric must we provide our machine (that is, function)? The answer, of course, will depend on the hour which we are providing fabric.

This leads us to the last type of function we will consider; it is known as an **inverse function** or, equivalently, as an **inverse mapping** or an **inverse transformation**. This is a very important type of function—as opposed to the regular definition of a function, $f\colon x \to f(x)$, an inverse function is denoted by f^{-1}, read "f-inverse," which is not to be confused with f raised to the power of -1. An inverse mapping is a mapping such that $f^{-1}\colon f(x) \to x$. It is worth noting that not all functions have inverses.

This definition can be rather confusing upon first glance. What it is telling us, though, is that there exists some function that takes our output result and gives us the necessary input. More literally, $f^{-1}\colon f(x) \to x$ can be read as "f-inverse maps the range $f(x)$ onto the domain x." The best way to understand any new concept is through an example, so we do that now.

Returning to our example, let's say that our function $g\colon t \to g(t)$ describes sweater-production. If we let $g = \frac{1}{2}t + 2$, where t is the hour and g has units of sweaters per hundred, then we wish to know g^{-1}—that is, for a certain amount of sweaters produced, $g(t)$, we want to know what hour these sweaters were produced. Do not get confused by the variable t; like x, it is only another way of expressing an unknown quantity. Luckily, there is an established procedure which we can follow to get g^{-1}.

The idea is that we replace every g with t and every t with g^{-1}. That is, we replace every function name with the variable, and every variable with our function-inverse. Then we simply solve for our function-inverse. In our case,

$$t = \frac{1}{2}g^{-1} + 2 \Longrightarrow g^{-1} = 2t - 4$$

Now, if we know our range, $\{y\} = g(t) = 2...8$. Applying our definition of an inverse function, then our range is now our domain, and our domain is now our range. So if we want to know in what hour 300 sweaters are produced, we compute $g^{-1}(t = 3)$, which is

$$g^{-1}(t = 3) = g^{-1}(3) = 2(3) - 4 = 6 - 4 = 2.$$

Then the hour in which 300 sweaters are produced is 2. We can verify this by computing $g(2)$. Let's do that, just to be sure:

$$g(2) = \frac{1}{2}(2) + 2 = 1 + 2 = 3$$

which verifies the above claim.

The final remark before we leave this section is that if we want to check that our inverse function is actually an inverse, we can do the operation

$$g\left(g^{-1}\right) = g^{-1}(g) = x,$$

where x is the variable. We check this with our previous example (remember that a function can have as its input something other than simply a number!).

$$g\left(g^{-1}\right) = \frac{1}{2}(2t - 4) + 2 = t - 2 + 2 = t$$

$$g^{-1}(g) = 2\left(\frac{1}{2}t + 2\right) - 4 = t + 4 - 4 = t$$

$$g\left(g^{-1}\right) = g^{-1}(g)$$

which shows that what we computed was, in fact, the correct inverse function.

1.3 What can we do with functions?

We have already seen that functions arise quite naturally in the physical world. Also, we have seen how to determine whether a given function is functional, into or onto, linear or nonlinear, and invertible. These are all very important characteristics of a function, and we will return to them again in the next section. For now, we want to know what operations we can perform on functions.

If we take our sweater-production function to be $g = \frac{1}{2}t + 2$ as we did in the previous section, and we now include fabric production to be defined by $f = 3t + 1$, where both g and f are expressed in one-hundred units of material per hour, then we want to know the total production at 2A.M. To do this, we add our functions together:

$$g + f = \left(\frac{1}{2}t + 2\right) + (3t + 1) = 3\frac{1}{2}t + 3$$

then plugging in $t = 2$, we get

$$f + g\,(t = 2) = f + g\,(2) = 3\frac{1}{2}\,(2) + 3 = 7 + 3 = 10,$$

so one-thousand units of material are produced at 2A.M. Indeed, we can do this for any time, t. Please note that

$$g + f = f + g$$

If we want to know how much fabric is excess (that is, how much fabric is not used in sweater production), then we can use subtraction,

$$g - f = \left(\frac{1}{2}t + 2\right) - (3t + 1) = -2\frac{1}{2}t + 1$$

For every hour, then, we can see that what we have is not an excess of fabric, but instead a *shortage* of fabric. We say this is a shortage because at every time, t, we have a negative value—if we had an excess, we would have a positive value. Please note that we subtracted fabric production from sweater production to get this result, and that, physically, it would not make sense to subtract sweater production from fabric production (what would we be calculating?). We verify below that what we have is, indeed, a shortage:

$$g - f\,(t = 3) = g - f\,(3) = -2\frac{1}{2}\,(3) + 1 = -7\frac{1}{2} + 1 = -6\frac{1}{2}$$

$$g - f\,(t = 8) = g - f\,(8) == 2\frac{1}{2}\,(8) + 1 = -20 + 1 = -19$$

You can convince yourself that as time goes to increasingly larger values, the negativity also increases. A reader might note that for negative values of t, we have an excess—however, it does not make much sense to have "negative" time, since time runs only in the forward direction. It is important to understand that unlike addition of functions,

$$f - g \neq g - f$$

Let's take another example: velocity. From physics, velocity is the rate of change of a particle's position in space. Thus velocity, v, is a function of both position, x (x is here a function, not a variable!), and time, t (t is a variable, not a function).

Let's say we are trying to compute the velocity (i.e. speed) of our car. We know our time domain is $\{t\} = 0...60$ seconds; however, we do not know our values of the position domain, $\{x\}$. We define v to be

$$v = 4x + 3t$$

If we know that our position function, x, which depends upon time, is given by,

$$x = 2.5t - 1$$

then we have two options for computing our values of v. We can compute x at a given time t and then plug in to our velocity function. You might suspect this kind of procedure can be extremely time-consuming. To remedy this, we have what is known as **function composition**. We have already seen this function-composition when we studied inverse mappings; however, we would now like to see it help us solve a real-world problem.

To do this, we want our velocity to be a function only of time. We simply "compose" our functions so that

$$v\left(x\left(t\right)\right) = 4\left(2.5t - 1\right) + 3t = 10t - 4 + 3t = 13t - 4$$

This function for v now depends only upon time. This type of function is known as an **Eulerian** function. On the other hand, a function which is time-independent is called **Lagrangian**. We can further compose two functions together in any way we like. For instance,

$$f = x + 4$$

and

$$g = 3x - 2$$

can be composed such that

$$f \circ g = f(g) = (3x - 2) + 4 = 3x + 2$$

or

$$g \circ f = g(f) = 3(x + 4) - 2 = 3x + 12 - 12 = 3x + 10$$

where $f \circ g$ is read as "f composed g" or "f composite g." Like subtraction,

$$f \circ g \neq g \circ f.$$

The usual rules for multiplication and division hold, as well. That is,

$$(f \cdot g) = f \cdot g = (x + 4) \cdot (3x - 2)$$

Note that

$$f \cdot g = g \cdot f.$$

For division,

$$(f \div g) = \frac{f}{g} = \frac{x + 4}{3x - 2}$$

which implies that

$$g \div f = \frac{g}{f} = \frac{3x - 2}{x + 4} \neq f \div g$$

Functions which are divided by other functions are called **rational functions**. These functions will play an increasingly important role throughout the course. One can see that the domain—that is, the values of x which a function can accept—can become limited whenever we have rational functions. For instance, one can see that problems arise in the previous example of a rational function when $x = -4$ because

$$\frac{g(-4)}{f(-4)} = \frac{3(-4) - 2}{(-4) + 4} = \frac{-14}{0}$$

and we cannot divide by zero. These kinds of problems will be considered from a geometric viewpoint later on in the course.

In conclusion, we can think of functions simply as name-tags for other equations. In the preceding examples, one will notice that all we ever did was replace a function by its corresponding equation. To this point, we have only done operations on linear functions; however, the techniques here are the same for the nonlinear case as well.

1.4 How can we solve nonlinear functions?

Nonlinear functions are studied so heavily, some of them even have special names. For instance, a function with a highest degree of 2 is called "quadratic," where

$$f(x) = ax^2 + bx + c \text{ , where a, b, and c are constants}$$

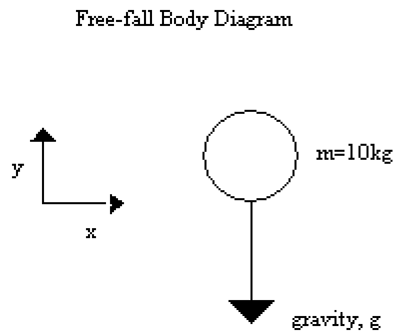
is the general form of a quadratic expression. Also,

$$f(x) = ax^3 + bx^2 + cx + d \text{ , where a, b, c, and d are constants}$$

is called a "cubic" function because the highest power appearing in the expression is 3. Likewise, when the highest appearing power in a function is 4, it is called quartic; when the highest power is 5, quintic. Once functions get beyond fifth-order terms, we do not really have names for them; this is because solution methods for functions of exceptionally high-order are not known.

For our purposes, we will contain ourselves to discussions of the quadratic function and special cubic and quartic functions which can be simplified back to quadratic (or linear!) equations. It is important to realize that, in general, functions which are quadratic or higher are not linear. When we deal with rational expressions later, we will have to reconsider linearity. Until that point, however, we can assume all functions of order higher than 1 are nonlinear.

The reasons for finding solutions to nonlinear functions are physically motivated here, and an elementary method of solving quadratic equations is presented. We begin by considering an object (namely, a ball) in free-fall. The ball has mass of 10kg and is held at rest above the ground before being released. The ball falls only under the effect of gravity. At the instant that the ball is 2.0 meters above the ground, the speed of the ball is 10.954m/s.



We want to know at what height the ball was initially released. Before looking to equations to help us understand the kinematics of this falling ball, we first need to examine the data we're given. Often in physics and engineering problems, looking to the data supplied will govern what equations you can choose to model the situation.

We know that

- y_0=unknown (initial height)
- v_0=0 (initial velocity)
- $y = 2$m (distance from initial height)
- $v = 10.954$m/s (velocity at 2m above the ground)
- $m = 10$kg (mass of the ball)
- $g = 10$ m/s^2

One thing to notice first of all is that this is an entirely Lagrangian description—there is no component of time or any dependence upon time acknowledged anywhere in the data. So we know that our governing function must also be Lagrangian (since if it were Eulerian, we could not solve for anything). Also, since our ball does not move to the left or right, it only moves along the straight line going up and down. This means our function must describe one-dimensional motion.

Luckily for us, there is one such equation,

$$v^2 = v_0^2 + 2g\left(y - y_0\right).$$

Note that we have replaced the x and x_0 with y and y_0 because we are dealing with vertical motion instead of horizontal motion. Also note that $v_0 = 0$, as we showed above. Since we are trying to solve for y_0, we isolate the y_0 term by dividing both sides by $2g$; that is,

$$\frac{v^2}{2g} = y - y_0$$

now we solve for y_0 and get,

$$y_0 = y + \frac{v^2}{2g}$$

The reason the term $\frac{v^2}{2g}$ is positive is because of the way we have defined our coordinate system. Plugging in our values yields

$$y_0 = 2 + \frac{(10.954)^2}{2(10)} = 6$$

Thus our ball was dropped from a height of 6m.

This example did not reveal much to us about the mathematics behind the physics; however, we now wish to ask a different question about this ball: at what time does it hit the ground? We define a function, $h\colon t \to h\,(t)$, that maps the height of the ball as a function of the time it is in the air. This mapping is expressed as

$$h(t) = \frac{t^2}{2} - 4t + y_0,$$

where h is a constant and is defined as the total height the ball must fall before reaching the ground. We can verify that this function does indeed map the time into the function by looking at the position of the ball at the initial time; that is,

$$h(t = 0) = h(0) = \frac{0^2}{2} - 3(0) + h = 0 - 0 + y_0 = y_0 = 6$$

which is indeed true.

Now, in order to find the time at which the ball reaches the ground, we must set $h(t) = 0$. This means that

$$0 = \frac{t^2}{2} - 4t + y_0$$

This equation can obviously not be solved by isolating our t. If we attempt to do so, we see that

$$\begin{aligned}
-y_0 &= \frac{t^2}{2} - 4t + y_0 - y_0 \\
-y_0 &= t\left(\frac{t}{2} - 4\right) \\
\Rightarrow t &= -\frac{y_0}{\frac{t}{2} - 4}
\end{aligned}$$

or else

$$\begin{aligned}
-y_0 &= \frac{t^2}{2} - 4t + y_0 - y_0 \\
-y_0 &= t\left(\frac{t}{2} - 4\right) \\
-\frac{y_0}{t} &= \left(\frac{t}{2} - 4\right) \\
\Rightarrow t &= 2\left(-\frac{y_0}{t} + 4\right)
\end{aligned}$$

neither of which isolate the t, since it is still on both sides of the equation.

We must introduce a new approach. We first write down our quadratic with the necessary conditions (i.e. $h(t) = 0$) imposed,

$$0 = \frac{t^2}{2} - 4t + 6.$$

The method we will use to solve this equation is known as **factoring**. Factoring is one of the most important skills you will learn in this Algebra II course, and it is important that you become very familiar with recognizing factorizable functions without much work. This involves lots of practice, and so in the Problems section of this chapter, there are a list of functions which can be factored to help give you an understand for what's going on.

In order to begin this factoring, we check to make sure that our function is in quadratic form,

$$ax^2 + bx + c = 0,$$

where $a = \frac{1}{2}$, $b = -4$ and $c = 6$, as we can see from our function. We make use of the following fact, called the **Zero Factor Property**:

$$a \cdot b \neq 0 \text{ unless either } a = 0 \text{ and/or } b = 0$$

Be careful though! Just because we might have

$$ab = 6 \text{ does NOT mean that either } a = 6 \text{ or } b = 6.$$

We could just as easily have $a = 3$ and $b = 2$, so don't misuse this fact.

Before solving any quadratic equation, we must make sure that all of our terms are on one side of the equals sign. This is important because to use the Zero Factor Property, we must have one side equal to zero. We have already done this in our physics example.

We do not assume any familiarity with factoring. We start with a simple example. To factor any number, n, in previous courses you have listed the numbers which, when multiplied together, recreate n. For instance, to factor the number 12, you might have

$$12 = (2)(6) \text{ or } 12 = (3)(4) \text{ which yield } 12 = (2)(2)(3),$$

and of course there are many more combinations that we did not show. However, what we did show is what is known as *completely factoring* a number; that is, factoring a number's factors until all we have left are prime numbers.

Quadratics and, more generally, polynomials can be treated in the same way. For instance, the equation

$$x^2 - 16 = (x - 4)(x + 4)$$

and likewise,

$$x^4 - 16 = \left(x^2 - 4\right)\left(x^2 + 4\right).$$

We can verify these factorization by the method known as **FOIL**, which states that we multiply the **F**irst terms, then we multiply our **O**uter terms and add them to the first terms, then we multiply our **I**nner terms and add them to our first and outer terns, and finally we multiply our **L**ast terms and add them to the first, outer, and inner terms in order to recreate the polynomial. Let's do that with the $x^2 - 16$ example.

Our first terms are x and x. Multiplying these two together yields:

$$(x - 4)(x + 4) = x \cdot x... = x^2...$$

Our outer terms are x (from $(x-4)$) and 4 (from $(x+4)$). Multiplying these two together and adding them yields:

$$(x - 4)(x + 4) = x^2 + 4 \cdot x... = x^2 + 4x...$$

Our inner terms are -4 (from $(x-4)$) and x (from $(x+4)$). Multiplying them together and adding yields:

$$(x-4)(x+4) = x^2 + 4x + (-4) \cdot x ... = x^2 + 4x - 4x ... = x^2 ...$$

Finally, our last terms are -4 (from $(x-4)$) and 4 (from $(x+4)$). Multiplying them together and adding yields:

$$(x-4)(x+4) = x^2 + (-4)(4) = x^2 - 16$$

Recall that we were trying to prove that

$$x^2 - 16 = (x-4)(x+4)$$

which we have just done.

Now, let's return to our free-falling body question. Remember that we had a ball falling with a height function defined as

$$h(t) = \frac{t^2}{2} - 4t + 6$$

and since we are trying to find our time at initial impact, we know that $h(t) = 0$, so

$$0 = \frac{t^2}{2} - 4t + 6.$$

This is an equation of the form that satisfies the zero factor property we stated above. In order to get rid of the fraction, we make use of the fundamental property of algebra and multiply both sides of the equation by 2,

$$(2)0 = (2)\left(\frac{t^2}{2} - 4t + 6.\right)$$

which yields

$$0 = t^2 - 8t + 12.$$

Now we factor this equation, and get

$$0 = (t-2)(t-6)$$

We divide both sides of the equation by $(t-2)$ and get,

$$0 = t - 6 \Rightarrow t = 6$$

which tells us that the ball hits the ground at $t = 6$s. However, we could have just as easily divided both sides by $(t - 6)$. Let's do that now; we see that this yields,

$$0 = t - 2 \Rightarrow t = 2,$$

so that we also have an impact at $t = 2$s. Which of these answers is the correct one? You may be surprised to know that *both* of these answers are correct. Unlike linear equations, which are always guaranteed to have only one solution, nonlinear equations can have more than one solutions, as we see here.

The reason for the two solutions is that—the ball bounces! The ball initially hits the floor at $t = 2$s (we know this because it would not make sense to say the ball hits the floor at $t = 6$s and then at 2s, since this would imply time is acting in reverse), bounces, and hits the floor again (and stops bouncing) at $t = 6$s. Thus our answer is 2s.

1.5 Problems